

# Effective Field Theory of Neutron Star Superfluidity

James Hormuzdiar\*

Department of Physics and Astronomy,  
Yale University, New Haven, CT 06520

Stephen D.H. Hsu<sup>†</sup>

Department of Physics,  
University of Oregon, Eugene OR 97403-5203

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## Abstract

We apply effective field theory and renormalization group techniques to the problem of Cooper pair formation in neutron stars. Simple analytical expressions for the  $^1S_0$  condensate are derived which are free of nuclear potential model dependencies. The condensate is evaluated using phase shift data from neutron-neutron scattering.

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\*james.hormuzdiar@yale.edu

<sup>†</sup>hsu@duende.uoregon.edu

Neutron superfluidity plays an important role in the physics of neutron stars, affecting the neutrino cooling rate and heat capacity as well as starquake phenomena [1]. However, the computation of the neutron-neutron (NN) condensate is a difficult problem and has resulted in a wide range of results [2, 3, 4, 5, 6]. The main reason is the exponential sensitivity of the condensate to the effective interaction, which is typically extracted in a model-dependent way from NN scattering data, and also depends on medium effects.

In this paper, we use the renormalization group (RG) to construct the effective field theory of neutrons near the Fermi surface. In this description the size of the condensate is related to the position of a Landau pole in a running coupling near the Fermi surface (FS) [7, 8]. All interactions other than the Cooper pairing interaction (a four-neutron operator restricted to kinematic points describing scattering of neutrons with equal and opposite momenta) can be shown to be irrelevant at the FS. Due to this simplification the evolution of the Cooper pairing interaction, and hence the location of the Landau pole, can be determined analytically. Medium effects are reflected in the RG evolution of Fermi liquid parameters such as the effective mass and four-neutron operator, whose initial form  $G(p^2)$  is obtained by matching to NN phase shifts via an exact expression for the in-medium NN amplitude. We focus on the  $^1S_0$  condensate, although our techniques can also be applied to study condensates of higher angular momentum.

We begin by reviewing the effective field theory description of Fermi liquids [7, 8]. In this description we make a guess as to the form of the effective theory close to the Fermi surface. The obvious guess based on the dynamics of non-relativistic systems is that the theory is one of weakly interacting fermions: these are the dressed “quasi-particles” of solid state physics language. We will henceforth refer to these effective degrees of freedom as “neutrons”, with the understanding that they could in principle be related to the bare neutrons in a complicated way. Rather than treating other degrees of freedom such as pions, deltas, etc., as propagating degrees of freedom we will integrate them out leaving a potentially infinite sum over non-local, higher dimension fermion operators. The effective Lagrangian is simply

$$\mathcal{L} = \bar{\psi}_i(i\rlap{\not{\partial}} + \mu\gamma_0 - m)\psi_i + \cdots, \quad (1)$$

where the ellipsis denote higher dimension interaction terms. Although we will eventually specialize to the non-relativistic limit, we begin with a relativistic formulation because it allows us to systematically track the corrections to that limit.

The chemical potential in (1) naturally breaks the  $O(3,1)$  invariance of space-time to  $O(3)$  and furthermore picks out momenta of order  $p_F = \sqrt{\mu^2 - m^2}$ . It is therefore natural to study the theory as we approach the Fermi surface in a Wilsonian sense. We parameterize

four momenta in the following fashion

$$p^\mu = (p^0, \vec{p}) = (k^0, \vec{k} + \vec{l}) \quad (2)$$

where  $\vec{k}$  lies on the Fermi surface ( $|\vec{k}| = p_F$ ) and  $\vec{l}$  is orthogonal to it. We study the Wilsonian effective theory of modes near the Fermi surface, with energy and momenta

$$|k_0|, |\vec{l}| < \Lambda \quad , \quad \Lambda \rightarrow 0 \quad . \quad (3)$$

While this type of RG scaling is somewhat unfamiliar, it actually corresponds to thinning out fermionic degrees of freedom according to their eigenvalues under the operator  $i\vec{\partial} + \mu\gamma_0 - m$ . It is easy to see that eigenspinors of this operator with eigenvalues  $\lambda_n : |\lambda_n| < \Lambda$  correspond to states satisfying (3). Consider an eigenspinor of the form

$$\psi_p = u(E, \vec{p}) e^{i(p \cdot x - p_0 t)} \quad , \quad (4)$$

where  $u(E, \vec{p})$  satisfies the usual momentum-space Dirac equation with  $E = \sqrt{p^2 + m^2} = m + \mu \pm \mathcal{O}(\Lambda)$ , and  $p_0 = m \pm \mathcal{O}(\Lambda)$ . Then by direct substitution we see that  $\psi_p$  satisfies

$$(i\vec{\partial} + \mu\gamma_0 - m) \psi_k = \mathcal{O}(\Lambda) \quad . \quad (5)$$

Thus, the RG flow towards the Fermi surface just corresponds to taking the cutoff on eigenvalues of  $i\vec{\partial} + \mu\gamma_0 - m$  to zero.

Which operators are relevant in this limit? For our guess to make sense the kinetic term for the fermions must be invariant when we scale energies and momenta,  $k_0 \rightarrow sk_0$  (or,  $t \rightarrow t/s$ ), and  $\vec{l} \rightarrow s\vec{l}$ , with  $s < 1$ . We must be careful to satisfy all the global symmetries of the theory. In particular, there is a spurious symmetry of (1) in which we treat  $\mu$  as the temporal component of a fictitious  $U(1)_B$  gauge boson. In other words, the combined transformations  $\psi \rightarrow e^{i\theta t} \psi$  and  $\mu \rightarrow \mu + \theta$  leave the lagrangian (1) invariant. From this, we deduce that time derivatives acting on  $\psi$  and factors of  $\mu$  may only enter the effective theory in the combination  $(\partial_0 + \mu)\gamma_0$ . This requires the kinetic term of our effective theory to be of the form

$$\mathcal{S}_{eff} = \int dt d^3p \bar{\psi} \left( (i\partial_t + \mu)\gamma_0 - \vec{p} \cdot \vec{\gamma} - m \right) \psi \quad , \quad (6)$$

where  $m$  is the effective neutron mass. In the Wilsonian RG scaling, we eliminate all modes with energy and momenta  $|k_0|, |l| > \Lambda$ , where  $\Lambda$  is our cutoff. As discussed above, on the remaining degrees of freedom the operator  $(i\partial_t + \mu)\gamma_0 - \vec{p} \cdot \vec{\gamma} - m \sim \mathcal{O}(\Lambda)$  and therefore scales like  $s$ . We deduce that for (6) to remain invariant,  $\psi$  must scale as  $s^{-1/2}$ . Now consider the four fermion operator

$$G \int dt d^3\vec{p}_1 d^3\vec{p}_2 d^3\vec{p}_3 d^3\vec{p}_4 \bar{\psi}\Gamma\psi \bar{\psi}\Gamma\psi \delta^3(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_4) \quad (7)$$

where  $\Gamma$  contains any Lorentz or flavor structure. Naively, for  $\vec{l}$  close to zero, the delta function does not scale, and (7) is irrelevant since it scales as  $s$ . Higher dimension operators with extra powers of the fields are clearly irrelevant as well. The only operators that survive are four-fermion operators satisfying the kinematic constraint  $\vec{p}_1 \simeq -\vec{p}_2$ , in which case the delta function becomes

$$\delta(\vec{l}_1 + \vec{l}_2 - \vec{l}_3 - \vec{l}_4) \quad (8)$$

and scales as  $s^{-1}$ . The resulting four fermion operator is marginal, and quantum effects must be considered to determine its relevance to dynamics at the FS.

We now study the RG flow of the marginal Cooper pairing operator. We are primarily interested in the  $^1S_0$  component of this operator, which will be seen to dominate the others near the FS. We first consider the  $^1S_0$  component by itself, and later consider corrections to this approximation that result from loops involving higher angular momentum components. It is easy to show that angular momentum conservation forbids such corrections so the  $^1S_0$  evolution is exact.

Let us briefly review the kinematics of neutron-neutron scattering at the Fermi surface in the center of mass frame. Let the incoming momentum be  $\vec{p}_1$ , and the outgoing momentum  $\vec{p}_2$ , both of magnitude  $p_F$ . The scattering amplitude is in general a function of the angle between these two vectors. The form factor of an operator describing this scattering process can be decomposed into components corresponding to projections onto different angular momentum eigenstates, or spherical harmonics  $Y_m^l(\theta, \phi)$ . Actually, since the amplitudes are independent of  $\phi$  only the  $m = 0$  components are required, and the projection just involves integration over  $\theta$ . The  $^1S_0$  component is obviously the component which is independent of  $\theta$ , and hence has a constant form factor.

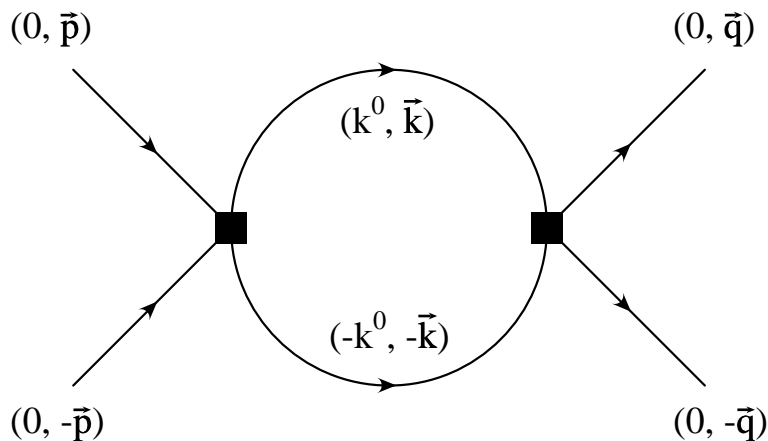


Figure 1: One loop renormalization of  $G$

The beta function for our four-neutron operator can be deduced from the one loop graph in figure 1. If we consider only the  $^1S_0$  component of the coupling the powers of  $G$  can be factored out of the integral, yielding<sup>1</sup>

$$-G^2 \int \frac{d^4k}{(2\pi)^4} \left[ \frac{i}{k^\mu \gamma_\mu + \mu \gamma_0 - m} \right]_{ik} \left[ \frac{i}{-k^\mu \gamma_\mu + \mu \gamma_0 - m} \right]_{jl} . \quad (9)$$

We can rewrite this as

$$G^2 \left[ -\frac{(\mu \gamma_0 + m)_{ik} (\mu \gamma_0 + m)_{jl}}{2} + \frac{1}{12} p_F^2 \sum_{a=1}^3 (\gamma_a)_{ik} (\gamma_a)_{jl} \right] I , \quad (10)$$

where the integral  $I$  is given by

$$\begin{aligned} I &= \frac{p_F}{\mu} \int \frac{d\Omega_k}{(2\pi)^3} \int \frac{dk_0 dl}{2\pi} \frac{1}{k_0^2 - l^2 - i\epsilon} \\ &= \frac{i p_F}{2\pi^2 \mu} \ln \left( \frac{\Lambda_{IR}}{\Lambda_{UV}} \right) . \end{aligned} \quad (11)$$

Here  $\Lambda_{IR}$  and  $\Lambda_{UV}$  are the infrared and ultraviolet limits of integration. In the usual Wilsonian sense, the effects of modes in the shell between these cutoffs is summarized in the evolution of the coupling  $G$ . In the non-relativistic limit, where  $\frac{p_F^2}{m^2} \rightarrow 0$ , (10) becomes

$$\delta G = G^2 \left[ \frac{\gamma_0 + 1}{2} \right]_{ik} \left[ \frac{\gamma_0 + 1}{2} \right]_{jl} \frac{m p_F}{2\pi^2} t \quad (12)$$

where  $t = \ln(\frac{\Lambda_{IR}}{\Lambda_{UV}})$ . To incorporate non-relativistic corrections to some order in  $p_F^2/m^2$ , one must include additional operators appearing in (11) in the RG equations. However, in our case of interest,  $p_F^2/m^2$  is at most of order a few percent, so we will drop all non-relativistic corrections. Since  $\frac{1}{2}(\gamma_0 \pm 1)$  is simply the neutron/antineutron projection operator, the result only renormalizes the coupling between neutrons, and does not involve anti-neutrons. Henceforth we will focus on this interaction. The resulting RG equation is

$$\frac{dG}{dt} = \frac{m p_F}{2\pi^2} G^2 . \quad (13)$$

The solution to this equation is

$$G(t) = \frac{2\pi^2 G(0)}{2\pi^2 - t m p_F G(0)} , \quad (14)$$

which has a Landau pole at

$$t = \frac{2\pi^2}{m p_F G(0)} . \quad (15)$$

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<sup>1</sup>Note that the  $i\epsilon$  prescription for propagators at finite chemical potential is slightly subtle [9]. Essentially, the sign of the  $i\epsilon$  terms is such that the usual Wick rotation can be made regardless of the sign of  $(E(\vec{p}) - \mu)$ .

By dimensional analysis, we expect a Cooper pairing gap of size

$$\Delta \simeq \Lambda_{UV} e^{\frac{2\pi^2}{m p_F G(0)}} \quad . \quad (16)$$

The exponent in (16) is very similar to the usual BCS weak-coupling result [2], except that in our case the coupling  $G$  has a well-defined origin: it arises from the matching of our purely nucleonic effective theory to the full theory and can be extracted from NN scattering data. The prefactor will be determined by matching to known results for the gap at low density.

Now we return to the issue of higher angular momentum components. As discussed previously we work in the basis provided by the spherical harmonics  $Y_m^l(\theta, \phi)$ . Actually we only require the  $m = 0$  harmonics, which are  $\phi$  independent. Breaking  $G(\theta)$  into its spherical harmonic components

$$G(\theta) = G^{(0)} Y_0^0(\theta, 0) + G^{(1)} Y_0^1(\theta, 0) + G^{(2)} Y_0^2(\theta, 0) + \dots \quad (17)$$

and repeating the previous analysis, the only change is in the replacement

$$G^2 \int d^2k \rightarrow \int d^2k G(\beta_{pk}) G(\beta_{kq}) \quad (18)$$

where  $\beta_{ab}$  is the angle between  $\vec{a}$  and  $\vec{b}$ .

Inserting (17) and simplifying yields the new set of RG equations

$$\frac{dG^{(l)}}{dt} = m p_F \sum_{l'=0}^{\infty} (G^{(l')})^2 \chi^{ll'} \quad (19)$$

where  $\chi^{ll'}$  is given in terms of integrals over Legendre polynomials. It is easy to show that  $\chi^{ll'}$  is diagonal, using the following result:

$$\int d^2k Y_0^l(\beta_{pk}, 0) Y_0^{l'}(\beta_{kq}, 0) \propto Y_0^l(\beta_{pq}, 0) \delta^{ll'} \quad . \quad (20)$$

This implies that there is no mixing between different angular momentum components, and our treatment of the  $^1S_0$  RGE is exact.

The final step in our analysis is to match the effective neutron Lagrangian to the full nuclear theory, which must include pions as well as neutron-neutron contact terms [10]. However, since our FS effective theory includes only very low energy quasiparticles, the pions have been integrated out. We need only retain the non-local four-neutron operator, whose value can be fixed by comparison with  $^1S_0$  NN scattering data.

First we note that the  $^1S_0$  scattering amplitude in the theory with only neutrons can be found by summing a bubble chain of Feynman graphs (figure 2) in which the vertices are given by the Lagrangian form factor  $G(p^2, \nu)$ . Here we work in the center of mass frame, so

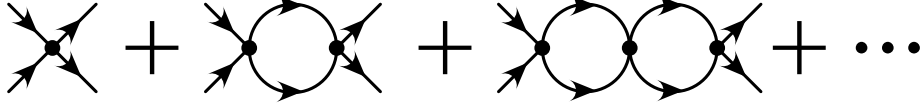


Figure 2: Bubble Sum leading to  $i\mathcal{M}$

$G$  is only a function of the neutron momentum  $p$ . We perform our calculation in an effective theory living in a thin shell around the Fermi surface, regulated by a hard ultraviolet cutoff  $\Lambda_{UV}$  and a hard infrared cutoff  $\Lambda_{IR}$ . The resulting amplitude can be directly computed and is given by

$$i\mathcal{M} = \frac{-iG(p^2, \nu)}{1 + m G(p^2, \nu) (\nu + ip)/4\pi} \quad , \quad (21)$$

where

$$\nu = \frac{2p}{\pi} \ln\left(\frac{\Lambda_{IR}}{\Lambda_{UV}}\right) \quad . \quad (22)$$

Invariance of the physical scattering amplitude  $\mathcal{M}$  under changes in  $\nu$  implies the following RGE for the coupling  $G(p^2, \nu)$ :

$$\frac{1}{G(p^2, \nu')} = \frac{1}{G(p^2, \nu)} + \frac{m(\nu - \nu')}{4\pi} \quad , \quad (23)$$

and a corresponding Landau pole at

$$\nu_* = \nu + \frac{4\pi}{mG(p^2, \nu)} \quad . \quad (24)$$

Comparison with (13) reveals that the two RGEs are identical. The scale associated with the Landau pole is

$$\Lambda_{IR}^* = \Lambda_{UV} e^{\pi\nu_*/2p} \quad . \quad (25)$$

Note that, unlike the scattering amplitude  $\mathcal{M}$ , the coupling  $G(p^2, \nu)$  is not a physical quantity and cannot be determined without fixing a subtraction scheme (i.e. specifying  $\nu$ ). This leads to an ambiguity in the overall normalization of the scale associated with the Landau pole.

We can rewrite (21) in terms of a phase shift, defined in terms of the S-matrix by  $S = e^{2i\delta}$ :

$$\delta = \frac{1}{2i} \ln \left( 1 + i \frac{m p}{2\pi} \mathcal{M} \right) \quad . \quad (26)$$

Finally, we can invert this relationship to obtain an expression for the coupling,

$$G(p^2, \nu) = -\frac{4\pi/m}{\nu + p \cot(\delta)} \quad . \quad (27)$$

Substituting this into (25) yields an equation relating the superfluid gap (which must be equal to the scale of the Landau pole, up to a some factor of order one) to the phase shift,

$$\Delta = \Lambda e^{-\frac{\pi}{2} \cot \delta} . \quad (28)$$

The constant  $\Lambda$  is undetermined due to the ambiguity mentioned above, but scales like the Fermi energy  $\epsilon_F = p_F^2/2m^*$ , since it is proportional to the UV cutoff or FS shell thickness. The precise numerical value of the coefficient can be determined by studying the weak coupling limit of a low-density neutron gas ( $p_F \rightarrow 0$ ). Once determined, the coefficient remains fixed independent of the Fermi momentum  $p_F$ . An explicit computation using the gap equation has been performed in this limit by Khodel et al. [6], with the result  $\Lambda = \frac{8}{e^2} \epsilon_F$ . The final result is

$$\Delta = \frac{8}{e^2} \frac{p_F^2}{2m_*} e^{-\frac{\pi}{2} \cot \delta} . \quad (29)$$

Our calculation can be viewed as a rigorous justification for the use of a BCS-like expression depending on Fermi liquid parameters which incorporate in-medium effects [2]. In our case the coefficient of the exponential can be justified from first principles. Equation (29) is exact when evaluated using the effective mass and in-medium phase shift data. If such data is not available, an estimate can be obtained by using the corresponding vacuum data, although as is shown in figure 3 the difference could be significant.

In figure 3 we display the resulting gap as a function of Fermi momentum  $p_F$ , obtained by inserting  $^1S_0$  phase shifts obtained from the Nijmegen partial wave analysis of NN scattering data [11]. The result of (29), using the bare neutron mass and zero-density (non-medium) phase shifts, is given by the solid curve. Inserting the effective mass and scattering amplitudes from [2] yields the short-dashed curve. The long-dashed and dot-dashed curves give the results of [2] (lower curve) and [6] (upper curve).

It appears that the result (29) is accurate as well as simple. Our results agree reasonably well with state of the art calculations using more traditional methods. Our leading result (ignoring medium corrections) is intermediate between calculations with “induced potential” effects [2, 3, 4], which have a maximum gap size of order 1 MeV, and those using a bare potential [4, 5, 6], which produce a maximum gap of approximately 3 MeV. When medium effects are included the gap is decreased significantly.

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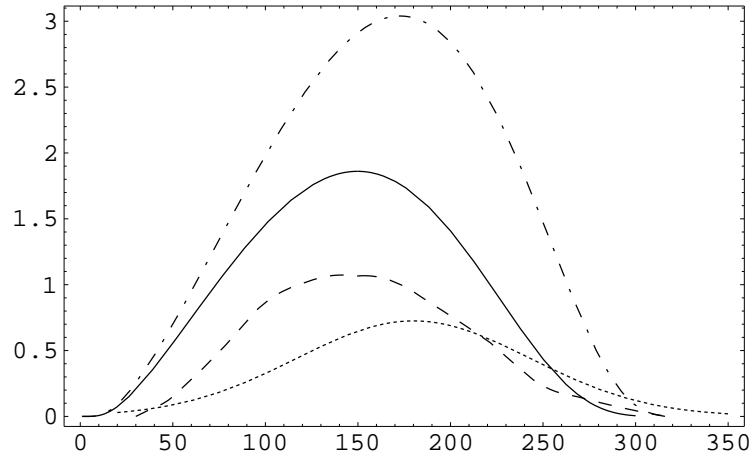


Figure 3: Delta vs. Fermi momentum  $p_F$ . All units are MeV. Solid curve represents Eq. (29). Long-dashed and dot-dashed curves are from [2] and [6]. Short-dashed curve is (29) with inputs from [2].

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